

Topos Quantum Theory on Quantization-Induced Sheaves

Kunji Nakayama*
Faculty of Law
Ryukoku University
Fushimi-ku, Kyoto 612-8577

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Abstract

In this paper, we construct a sheaf-based topos quantum theory. It is well known that a topos quantum theory can be constructed on the topos of presheaves on the category of commutative von Neumann algebras of bounded operators on a Hilbert space. Also, it is already known that quantization naturally induces a Lawvere-Tierney topology on the presheaf topos. We show that a topos quantum theory akin to the presheaf-based one can be constructed on sheaves defined by the quantization-induced Lawvere-Tierney topology. That is, starting from the spectral sheaf as a state space of a given quantum system, we construct sheaf-based expressions of physical propositions and truth objects, and thereby give a method of truth-value assignment to the propositions. Furthermore, we clarify the relationship to the presheaf-based quantum theory. We give translation rules between the sheaf-based ingredients and the corresponding presheaf-based ones. The translation rules have ‘coarse-graining’ effects on the spaces of the presheaf-based ingredients; a lot of different proposition presheaves, truth presheaves, and presheaf-based truth-values are translated to a proposition sheaf, a truth sheaf, and a sheaf-based truth-value, respectively. We examine the extent of the coarse-graining made by translation.

*e-mail: nakayama@law.ryukoku.ac.jp

1 Introduction

Since Isham [1] applied topos theory to history quantum theory, topos theoretic approach to quantum theory has been studied by many researchers [2–18]. In this approach, quantum theory is reformulated within a framework of intuitionistic (hence, multi-valued) logic. Every physical proposition about a given quantum system is assigned a truth-value without falling foul of the Kochen-Specker no go theorem [19]. Therefore, the topos approach permits some kind of realistic interpretation regarding values of physical quantities that does not require things like the notion of measurement. Because of this, it can provide a promising framework for quantum gravity theory and quantum cosmology.

There are a few different ways of topos approach. Among them, we focus on the formalism made by Döring and Isham [6–9, 14, 15]. They adopted the topos of presheaves on the category of commutative von Neumann algebras of bounded operators on a Hilbert space. In their theory, the spectral presheaf plays a key role similar to state space of classical physics. As a result of the Kochen-Specker theorem, we cannot assign to every physical quantity of a quantum system a sharply determined value, which means there are no global elements of the spectral presheaf [2–5]. In this respect, the spectral presheaf is largely different from the state space of classical physics, since the latter consists of points, each of which corresponds to a state where every physical quantity has sharply determined value. Nonetheless, the spectral presheaf can work as a state space in that every physical proposition about a given quantum system can be expressed as its subobject, as every physical proposition about a classical system can be identified with its extensional expression, i.e., a subset of state space. By regarding the spectral presheaf as state space and using it in a topos theoretical framework, Döring and Isham succeeded in giving a method that assigns to every physical proposition a truth-value.

Döring and Isham’s theory is an abstract, general theory; it does not need to be related to concrete classical systems, like ordinary quantum theories that are axiomatically or algebraically formulated on Hilbert spaces or C^* -algebras. This is the case for the other topos quantum theories obtained so far. If quantization of a classical system is taken into consideration, however, some extra structures are induced on the topos on which a quantum theory is formulated. In fact, Nakayama [20] showed that quantization that is given by a function from classical observables to self-adjoint operators on a Hilbert

space naturally induces a Lawvere-Tierney topology on the presheaf topos of Döring and Isham. It is well-known that any Lawvere-Tierney topology defines sheaves, and furthermore, the collection of all such sheaves also forms a topos [21]. Thus, from the presheaf topos, we obtain another topos consisting of sheaves via quantization.

One question would arise. Can we construct a quantum theory on the topos of quantization-induced sheaves? One of the purposes of the present paper is to give an affirmative answer to the question. We can construct a topos quantum theory on the quantization-induced sheaves in a way akin to the presheaf-based theory of Döring and Isham. Such a theory could be canonical as a theory of the system quantized from the classical one since quantum observables corresponding to classical ones are identified therein.

Furthermore, the theory on quantization-induced sheaves can be formulated by means of topos-theoretic ingredients smaller than those of the presheaf-based theory. For example, as we will see, the space of truth-values of the quantization-induced topos is smaller than that of the matrical topos of presheaves. This is because, for each sheaf-based truth-value, there are a lot of different presheaf-based ones that can be regarded as its ‘translations’, and conversely, a lot of different presheaf-based truth-values are translated to one and the same sheaf-based one. The same holds for the space of propositions and that of truth objects, because each sheaf-based proposition and each truth object have a lot of different presheaf-based translations. We call these properties coarse-graining made by translation.

Another question would arise. To what degree do the spaces of presheaf-based truth-values, propositions, and truth objects get coarse-grained via translation? In this paper, we answer this question to some extent. We give translation rules between the sheaf-based ingredients and the presheaf-based ones, and for an arbitrarily given sheaf-based one, we explicitly construct corresponding subspaces consisting of its presheaf-based translations that are regarded as the same from the sheaf-based viewpoint.

The present paper is organized as follows. In section 2, we briefly review Nakayama’s result [20] about quantization-induced topologies and sheaves. Further, additional explanation about some related notions that we will need in later sections are given. In section 3, we develop the sheaf-based method of truth-value assignment. This is done along the line of the presheaf-based method given by Döring and Isham [15], which we briefly summarize in appendix A for referential convenience. (We should, however, note that the main purpose of Döring and Isham [15] is not to give the method itself but

to propose a new interpretation for quantum probabilities, which is beyond the scope of the present paper.) In section 4, we give rules of translation of the ingredients necessary for truth-value assignment between the sheaf-based and the presheaf-based cases. In section 5, we deal with the coarse-graining problem mentioned above. Main results obtained therein are presented by theorems 5.1, 5.5, and 5.7.

2 Topos of Sheaves Induced by Quantization

In this section, we give a brief review of the results by Nakayama [20] and some supplementary explanations. Nakayama [20] defines quantization as an injective map v from a Lie algebra \mathcal{O} , a model of classical observables [22], to self-adjoint operators on a Hilbert space \mathcal{H} . The quantization map naturally defines a functor ϕ from the category $\mathbf{C}(\mathcal{O})$ of sets of commutative classical observables to the category \mathbf{V} of commutative von Neumann algebras of bounded operators on \mathcal{H} . The functor ϕ assigns to each $C \in \mathbf{C}(\mathcal{O})$ the least commutative von Neumann algebra that includes $e^{iv(C)}$. We define a functor $\psi : \mathbf{V} \rightarrow \mathbf{C}(\mathcal{O})$ by

$$\psi(V) := \{a \in \mathcal{O} \mid e^{iv(a)} \in V\}. \quad (2.1)$$

The functors ϕ and ψ give a Galois connection between $\mathbf{C}(\mathcal{O})$ and \mathbf{V} .

The endofunctor $\flat := \phi\psi : \mathbf{V} \rightarrow \mathbf{V}$ induces a Grothendieck topology J on \mathbf{V} , which is defined by for each $V \in \mathbf{V}$,

$$J(V) := \{\omega \in \Omega(V) \mid \flat(V) \in \omega\}, \quad (2.2)$$

where Ω is the subobject classifier of the topos $\widehat{\mathbf{V}} \equiv \mathbf{Set}^{\mathbf{V}^{\text{op}}}$ of presheaves on \mathbf{V} .

As is well-known, every Grothendieck topology on \mathbf{V} is equivalent to a Lawvere-Tierney topology on $\widehat{\mathbf{V}}$. (As for general theory of topoi, see e.g., MacLane and Moerdijk's textbook [21].) The Grothendieck topology (2.2) gives the Lawvere-Tierney topology $\Omega \xrightarrow{j} \Omega$ defined by, for each $V \in \mathbf{V}$ and $\omega \in \Omega(V)$,

$$j_V(\omega) := \{V' \subseteq_{\mathbf{V}} V \mid \flat(V') \in \omega\}, \quad (2.3)$$

where $V' \subseteq_{\mathbf{V}} V$ means that $V', V \in \mathbf{V}$ and $V' \subseteq V$.

Each Lawvere-Tierney topology is equivalent to a closure operator. In the present case given by (2.3), for each presheaf $Q \in \widehat{\mathbf{V}}$ and its subobject

$S \in \text{Sub}(Q)$, the closure \bar{S} of S in Q is defined by

$$\bar{S}(V) := \{q \in Q(V) \mid Q(\flat(V) \hookrightarrow V)(q) \in S(\flat(V))\}. \quad (2.4)$$

Any Lawvere-Tierney topology j on $\widehat{\mathbf{V}}$ defines sheaves as follows: Let $S \in \text{Sub}(Q)$ be dense in Q , that is, $\bar{S} = Q$. Then, a presheaf R is called a sheaf associated with a topology j , or simply, j -sheaf, if and only if, for any morphism $\lambda \in \text{Hom}(S, R)$, there exists one and only one morphism $\mu \in \text{Hom}(Q, R)$ that makes the diagram

$$\begin{array}{ccc} S & \xrightarrow{\lambda} & R \\ \text{dense} \downarrow & \nearrow \mu & \\ Q & & \end{array} \quad (2.5)$$

commute. All j -sheaves and all morphisms between them form a topos, which is denoted by $\text{Sh}_j \widehat{\mathbf{V}}$.

Sheaves associated with the topology (2.3) are expressed by the functor $\flat^* : \widehat{\mathbf{V}} \rightarrow \widehat{\mathbf{V}}$ that is defined by, for each $Q \in \widehat{\mathbf{V}}$,

$$(\flat^* Q)(V) := Q(\flat(V)), \quad (2.6)$$

and for any $V' \subseteq_{\mathbf{V}} V$,

$$(\flat^* Q)(V' \hookrightarrow V) := Q(\flat(V') \hookrightarrow \flat(V)). \quad (2.7)$$

We can show that a presheaf Q is a j -sheaf if and only if Q is isomorphic to $\flat^* Q$. To make the condition more precise, we define a morphism $Q \xrightarrow{\zeta_Q} \flat^* Q$ by $(\zeta_Q)_V := Q(\flat(V) \hookrightarrow V)$. Then, Q is a j -sheaf if and only if ζ_Q is isomorphic. We should note that ζ_Q is natural with respect to $Q \in \widehat{\mathbf{V}}$. That is, ζ is a natural transformation from the identity functor $I : \widehat{\mathbf{V}} \rightarrow \widehat{\mathbf{V}}$ to the functor $\flat^* : \widehat{\mathbf{V}} \rightarrow \widehat{\mathbf{V}}$. Furthermore, we should note that \flat^* is in fact an associated sheaf functor (a sheafification functor) from $\widehat{\mathbf{V}}$ to $\text{Sh}_j \widehat{\mathbf{V}}$.

Returning to the diagram (2.5), we note that the morphism μ is given by

$$\mu = \zeta_Q^{-1} \circ \flat^* \lambda \circ \zeta_Q, \quad (2.8)$$

since the naturality of ζ makes the diagram

$$\begin{array}{ccccc}
 S & & \xrightarrow{\lambda} & & R \\
 \downarrow \zeta_S & \searrow \text{dense} & & \nearrow \mu & \downarrow \zeta_R \\
 & Q & & & \\
 & \downarrow \zeta_Q & & & \\
 & b^*Q & & & \\
 b^*S & \nearrow & & \searrow b^*\mu = b^*\lambda & b^*R \\
 & b^*\lambda & & &
 \end{array}
 \tag{2.9}$$

commute. Here, this digram reflects the fact that $\flat^*S = \flat^*\bar{S} = \flat^*Q$.

In our formalism, truth-values of physical propositions are taken on the subobject classifier Ω_j of $\mathbf{Sh}_j\widehat{\mathbf{V}}$. That is, they are given as global elements $1 \mapsto \Omega_j \in \Gamma\Omega_j$ of Ω_j . As is well-known, Ω_j is the equalizer of $\Omega \xrightarrow{1_\Omega} \Omega$ and $\Omega \xrightarrow{j} \Omega$. In the present case, Ω_j is a subobject of Ω given by

$$\Omega_j(V) := \{\omega \in \Omega(V) \mid \forall V' \subseteq_{\mathbf{V}} V \ (\mathfrak{b}(V') \in \omega \Rightarrow V' \in \omega)\}. \quad (2.10)$$

Since for each $V \in \mathbf{V}$, $\Omega_j(V)$ contains the set \mathbf{t}_V of all subalgebras of V as the top element, the truth arrow $\text{true}_j \in \Gamma\Omega_j$ is given by

$$(\text{true}_j)_V := \mathbf{t}_V. \quad (2.11)$$

Later, we will deal with power objects in $\mathrm{Sh}_j\widehat{\mathbf{V}}$. As is well-known, the power object $\mathbb{P}_j R \equiv \Omega_j^R$ of a j -sheaf R can be calculated in $\widehat{\mathbf{V}}$. That is, for each $V \in \mathbf{V}$,

$$\begin{aligned} (\mathbb{P}_j R)(V) &= \mathrm{Hom}(R_{\downarrow V}, (\Omega_j)_{\downarrow V}) \\ &\simeq \mathrm{Hom}(R_{\downarrow V}, \Omega_j), \end{aligned} \quad (2.12)$$

where $R_{\downarrow V}$ and $(\Omega_j)_{\downarrow V}$ are downward restrictions as presheaves, the definition of which is given by (A.4) and (A.5). (Since $\mathrm{Sh}_j \widehat{\mathbf{V}}$ is a full subcategory of $\widehat{\mathbf{V}}$, $\mathrm{Hom}_{\mathrm{Sh}_j \widehat{\mathbf{V}}}(A, B) = \mathrm{Hom}_{\widehat{\mathbf{V}}}(A, B)$ for arbitrary sheaves A and B . So we simply write $\mathrm{Hom}(A, B)$ for both of them omitting the subscripts $\mathrm{Sh}_j \widehat{\mathbf{V}}$ and

$\widehat{\mathbf{V}}.$) Also, for $V' \subseteq_{\mathbf{V}} V$ and $\lambda \in (\mathbb{P}_j R)(V)$, $\lambda|_{V'} \equiv (\mathbb{P}_j R)(V' \hookrightarrow V)(\lambda)$ is defined as the morphism that makes the diagram

$$\begin{array}{ccc} R_{\downarrow V'} & \xrightarrow{\lambda|_{V'}} & \Omega_j \\ \downarrow & \nearrow \lambda & \\ R_{\downarrow V} & & \end{array} \quad (2.13)$$

commute.

In order to give another, more useful expression of the power object $\mathbb{P}_j R$, we note that it is a sheaf representing the collection $\text{Sub}_j(R)$ of all subsheaves of R . Let Q be a presheaf. As we will see below, $\mathbb{P}_j(b^*Q)$ can be expressed as

$$\mathbb{P}_j(b^*Q)(V) \simeq \text{Sub}_j(b^*(Q_{\downarrow V})) \quad (2.14)$$

and

$$\mathbb{P}_j(b^*Q)(V' \hookrightarrow V) : \text{Sub}_j(b^*(Q_{\downarrow V})) \rightarrow \text{Sub}_j(b^*(Q_{\downarrow V'})); S \mapsto b^*(S_{\downarrow V'}). \quad (2.15)$$

In particular, since any j -sheaf R satisfies $R \simeq b^*R$, we have

$$\mathbb{P}_j R \simeq \mathbb{P}_j(b^*R) \simeq \text{Sub}_j(b^*(R_{\downarrow V})). \quad (2.16)$$

Expression (2.14) comes from the fact that

$$\begin{aligned} \mathbb{P}_j(b^*Q)(V) &\simeq \text{Hom}((b^*Q)_{\downarrow V}, \Omega_j) \\ &\simeq \text{Hom}(b^*(Q_{\downarrow V}), \Omega_j) \\ &\simeq \text{Sub}_j(b^*(Q_{\downarrow V})). \end{aligned} \quad (2.17)$$

Here, the bijectivity between the first and second lines on (2.17) is verified from the commutative diagram

$$\begin{array}{ccccc} (b^*Q)_{\downarrow V'} & \xrightarrow{\quad} & (b^*Q)_{\downarrow V} & & \\ \downarrow \text{dense} & \searrow \kappa|_{V'} & \swarrow \kappa & \downarrow \text{dense} & \\ & \Omega_j & & & \\ \swarrow \chi|_{V'} & & \searrow \chi & & \\ b^*(Q_{\downarrow V'}) & \xrightarrow{\quad} & b^*(Q_{\downarrow V}) & & \end{array} \quad (2.18)$$

That is, since Ω_j is a j -sheaf, and since, as easily shown, $(b^*Q)_{\downarrow V}$ is dense in $b^*(Q_{\downarrow V})$, χ is uniquely determined by use of (2.8) for each morphism κ .

To see consistency between (2.13) and (2.15), let S^χ be a subsheaf of $b^*(Q_{\downarrow V})$ corresponding to the characteristic morphism χ . Then, in the diagram

$$\begin{array}{ccc}
 b^*(S_{\downarrow V'}^\chi) & \xrightarrow{\quad} & S^\chi \\
 \downarrow & \swarrow \text{!} \quad \searrow \text{!} & \downarrow \\
 & 1 & \\
 & \downarrow \text{true}_j & \\
 & \Omega_j & \\
 \swarrow \chi|_{V'} \quad \searrow \chi & & \\
 b^*(Q_{\downarrow V'}) & \xrightarrow{\quad} & b^*(Q_{\downarrow V}),
 \end{array} \tag{2.19}$$

the trapezoid at the right hand side is a pullback, and so is the outer square as easily shown. Thus, also the trapezoid at the left hand side is a pullback, which means that $b^*(S_{\downarrow V'}^\chi)$ is identified as the subsheaf of $b^*(Q_{\downarrow V'})$ that corresponds to the characteristic morphism $\chi|_{V'} \equiv \mathbb{P}_j(b^*Q)(V' \hookrightarrow V)(\chi)$.

3 Truth-Value Assignment on Quantization-Induced Sheaves

In the theory of Döring and Isham [6–9, 14, 15], the spectral presheaf Σ , the definition of which is given in appendix A, plays a role of state space of a given quantum system. Every physical proposition P is assumed to be representable as a clopen subobject of Σ , that is, an element of the collection $\text{Sub}_{\text{cl}}(\Sigma)$ of all clopen subobjects of Σ . For instance, Döring and Isham showed that each projection operator \hat{P} , which corresponds to some physical propositions in ordinary quantum theory, naturally defines a clopen subobject $\delta(\hat{P})$ of Σ via the ‘daseinization operator’ δ . If we are given a quantum state, we can specify propositions regarded as true. They are represented by a truth object \mathbb{T} , of which global elements give the truth propositions. If we have \mathbb{T} , we can assign to every proposition P a truth value via topos-theoretical setting. In appendix A, we give a brief explanation of the method of truth-value assignment developed by Döring and Isham [15], the style of

which is helpful for us to construct a sheaf-based theory. (It should be emphasized, however, that the main purpose of [15] is not to give the valuation method summarize in appendix A, but to propose a new interpretation of quantum probabilities based on intuitionistic logic, which is beyond the scope of the purpose of the present paper.)

In our formalism, we appropriate the ‘spectral sheaf’ $\flat^*\Sigma$ for the role of state space. Namely, every proposition is assumed to be representable as a clopen subsheaf of $\flat^*\Sigma$. We, thus, regard $\text{Sub}_{j\text{cl}}(\flat^*\Sigma)$, the collection of all clopen subsheaves of $\flat^*\Sigma$, as a proposition space. It can be internalized to $\text{Sh}_j\widehat{\mathbf{V}}$ as a subsheaf $\mathbb{P}_{j\text{cl}}(\flat^*\Sigma)$ of $\mathbb{P}_j(\flat^*\Sigma)$ that is defined by

$$(\mathbb{P}_{j\text{cl}}(\flat^*\Sigma))(V) := \text{Sub}_{j\text{cl}}(\flat^*(\Sigma_{\downarrow V})). \quad (3.1)$$

This definition really gives a presheaf because $(\mathbb{P}_{j\text{cl}}(\flat^*\Sigma))(V' \hookrightarrow V)$, i.e., the restriction of $(\mathbb{P}_j(\flat^*\Sigma))(V' \hookrightarrow V)$ to $\text{Sub}_{j\text{cl}}(\flat^*(\Sigma_{\downarrow V}))$, takes values on $\text{Sub}_{j\text{cl}}(\flat^*(\Sigma_{\downarrow V'}))$. Furthermore,

Proposition 3.1 *The presheaf $\mathbb{P}_{j\text{cl}}(\flat^*\Sigma)$ is a j -sheaf.*

Proof. We have

$$\begin{aligned} \flat^*(\mathbb{P}_{j\text{cl}}(\flat^*\Sigma))(V) &= (\mathbb{P}_{j\text{cl}}(\flat^*\Sigma))(\flat(V)) \\ &= \text{Sub}_{j\text{cl}}(\flat^*(\Sigma_{\downarrow \flat(V)})) \\ &= \text{Sub}_{j\text{cl}}(\flat^*(\Sigma_{\downarrow V})) \\ &= (\mathbb{P}_{j\text{cl}}(\flat^*\Sigma))(V), \end{aligned} \quad (3.2)$$

where from the second line to the third, we used (B.3). Furthermore, for each $S \in \text{Sub}_{j\text{cl}}(\flat^*(\Sigma_{\downarrow V}))$,

$$\begin{aligned} (\zeta_{\mathbb{P}_{j\text{cl}}(\flat^*\Sigma)})_V(S) &= \flat^*(S_{\downarrow \flat(V)}) \\ &= \flat^*(S_{\downarrow V}) \\ &= S. \end{aligned} \quad (3.3)$$

Therefore, $\zeta_{\mathbb{P}_{j\text{cl}}(\flat^*\Sigma)}$ is a natural isomorphism. \square

As is well-known, $\text{Sub}_j(\flat^*\Sigma) \simeq \Gamma(\mathbb{P}_j(\flat^*\Sigma)) := \text{Hom}(1, \mathbb{P}_j(\flat^*\Sigma))$. That is, every $S \in \text{Sub}_j(\flat^*\Sigma)$ has its name $\lceil S \rceil_j \in \Gamma(\mathbb{P}_j(\flat^*\Sigma))$ defined by

$$(\lceil S \rceil_j)_V := \flat^*(S_{\downarrow V}), \quad (3.4)$$

and every $s \in \Gamma(\mathbb{P}_j(\flat^*\Sigma))$ has its inverse, i.e., the subsheaf $\lceil s \rceil_j^{-1}$ of $\flat^*\Sigma$ given by

$$\lceil s \rceil_j^{-1}(V) := (s_V)(V). \quad (3.5)$$

It is obvious that, for any $S \in \text{Sub}_j(\flat^*(\Sigma_{\downarrow V}))$, $\lceil S \rceil_j \in \Gamma(\mathbb{P}_{j\text{cl}}(\flat^*\Sigma))$ if and only if S is a proposition, i.e., $S \in \text{Sub}_{j\text{cl}}(\flat^*(\Sigma_{\downarrow V}))$. Furthermore, for each proposition $P \in \text{Sub}_{j\text{cl}}(\flat^*\Sigma)$, the diagram

$$\begin{array}{ccc} 1 & \xrightarrow{\lceil P \rceil_j} & \mathbb{P}_{j\text{cl}}(\flat^*\Sigma) \\ & \searrow \lceil P \rceil_j & \downarrow \\ & & \mathbb{P}_j(\flat^*\Sigma) \end{array} \quad (3.6)$$

commutes. Therefore, $\mathbb{P}_{j\text{cl}}(\flat^*\Sigma)$ is a canonical internalization of $\text{Sub}_{j\text{cl}}(\flat^*\Sigma)$.

We can express propositions in different ways. To do so, we need to invoke the outer presheaf O of Döring and Isham [7, 14, 15] and a few related notions. (As for the definition of O , see (A.6) and (A.7).) We call the sheafification \flat^*O of O the outer sheaf. Furthermore, we call a set $h \equiv \{\hat{h}_V \in (\flat^*O)(V)\}_{V \in \mathbf{V}}$ a hyper-element of \flat^*O , if

$$\hat{h}_V = \hat{h}_{\flat(V)} \quad \text{and} \quad \flat^*O(V' \hookrightarrow V)(\hat{h}_V) \preceq \hat{h}_{V'}. \quad (3.7)$$

This is a j -sheaf counterpart of the notion of hyper-elements (A.12) defined by Döring and Isham [15]. We write $\text{Hyp}_j(\flat^*O)$ for the collection of all hyper-elements of \flat^*O . Let $\text{Sub}_{j\text{dB}}(\flat^*O)$ be the collection of all downward closed, Boolean subsheaves of \flat^*O . That is, for all $P \in \text{Sub}_j(\flat^*O)$, $P \in \text{Sub}_{j\text{dB}}(\flat^*O)$ if and only if, for any $V \in \mathbf{V}$, $P(V)$ is a downward closed set of $(\flat^*O)(V)$ containing a top element. (Obviously, such $P(V)$'s are complete Boolean lattices.) We can regard $\text{Hyp}_j(\flat^*O)$ and $\text{Sub}_{j\text{dB}}(\flat^*O)$ as proposition spaces equivalent to $\text{Sub}_{j\text{cl}}(\flat^*\Sigma)$. This is because, corresponding to relation (A.10) proved by Döring and Isham [15], the following relation holds:

$$\text{Sub}_{j\text{dB}}(\flat^*O) \simeq \text{Hyp}_j(\flat^*O) \simeq \text{Sub}_{j\text{cl}}(\flat^*\Sigma). \quad (3.8)$$

Here, the bijection at the left hand side of (3.8) is realized by a function $c_j : \text{Sub}_{j\text{dB}}(\flat^*O) \xrightarrow{\sim} \text{Hyp}_j(\flat^*O)$ that is defined by

$$(c_j(A))_V := \vee A(V). \quad (3.9)$$

To see the right hand side of (3.8), we use the bijections $\alpha_V : O(V) \rightarrow \mathcal{Cl}(\Sigma(V))$ ($V \in \mathbf{V}$) introduced by Döring and Isham [15]. (For the definition, see (A.15).) These bijections allow us to regard $\{\mathcal{Cl}(\Sigma(V))\}_{V \in \mathbf{V}}$ as a presheaf $\mathcal{Cl}\Sigma$ isomorphic to the outer presheaf O , and $\{\alpha_V\}_{V \in \mathbf{V}}$ as a natural isomorphism $\alpha : O \xrightarrow{\sim} \mathcal{Cl}\Sigma$. Furthermore, α induces a natural isomorphism $\flat^*\alpha : \flat^*O \xrightarrow{\sim} \flat^*(\mathcal{Cl}\Sigma)$, where $\flat^*(\mathcal{Cl}\Sigma)(V) = \mathcal{Cl}\Sigma(\flat(V)) = \mathcal{Cl}(\Sigma(\flat(V))) = \mathcal{Cl}(\flat^*\Sigma(V))$. Therefore, we obtain a bijection $k_j : \text{Hyp}_j(\flat^*O) \xrightarrow{\sim} \text{Sub}_{j\text{cl}}(\flat^*\Sigma)$ that is given by

$$\begin{aligned} (k_j(h))(V) &:= (\flat^*\alpha)_V(\hat{h}_V) \\ &= \{\sigma \in (\flat^*\Sigma)(V) \mid \sigma(\hat{h}_V) = 1\} \\ &= \{\sigma \in \Sigma(\flat(V)) \mid \sigma(\hat{h}_{\flat(V)}) = 1\}, \end{aligned} \quad (3.10)$$

and hence, a bijection $f_j : \text{Sub}_{j\text{dB}}(\flat^*O) \xrightarrow{\sim} \text{Sub}_{j\text{cl}}(\flat^*\Sigma)$ defined by

$$(f_j(A))(V) := (\flat^*\alpha)_V(\vee A(V)). \quad (3.11)$$

It is obvious from (3.8) that $k_{j\downarrow V}$, $c_{j\downarrow V}$, and $f_{j\downarrow V}$, the restrictions of k_j , c_j , and f_j , respectively, to subalgebras of V , give the relation

$$\text{Sub}_{j\text{dB}}(\flat^*(O_{\downarrow V})) \simeq \text{Hyp}_j(\flat^*(O_{\downarrow V})) \simeq \text{Sub}_{j\text{cl}}(\flat^*(\Sigma_{\downarrow V})). \quad (3.12)$$

Therefore, the proposition space $\text{Sub}_{j\text{cl}}(\flat^*\Sigma) \simeq \text{Sub}_{j\text{dB}}(\flat^*O)$ can be internalized also as a subsheaf $\mathbb{P}_{j\text{dB}}(\flat^*O)$ of $\mathbb{P}_j(\flat^*O)$ that is defined by

$$(\mathbb{P}_{j\text{dB}}(\flat^*O))(V) := \text{Sub}_{j\text{dB}}(\flat^*(O_{\downarrow V})). \quad (3.13)$$

Every proposition $P \in \text{Sub}_{j\text{dB}}(\flat^*O)$ has its name $\lceil P \rceil_j \in \Gamma(\mathbb{P}_{j\text{dB}}(\flat^*O))$ in $\text{Sh}_j \hat{\mathbf{V}}$, which is given by $(\lceil P \rceil_j)_V := \flat^*(P_{\downarrow V})$.

The daseinization operator δ introduced by Döring and Isham [6, 7, 14, 15] assigns to each projection operator \hat{P} on \mathcal{H} a global element $\delta(\hat{P})$ of the outer presheaf O . (For the definition, see (A.8).) As a counterpart of δ , we introduce a map δ_j , which assigns to each \hat{P} a global element of \flat^*O by

$$\begin{aligned} \delta_j(\hat{P})_V &:= \bigwedge \{\hat{a} \in (\flat^*O)(V) \mid \hat{P} \preceq \hat{a}\} \\ &= \bigwedge \{\hat{a} \in O(\flat(V)) \mid \hat{P} \preceq \hat{a}\} \\ &= \delta(\hat{P})_{\flat(V)}. \end{aligned} \quad (3.14)$$

To see that really $\delta_j(\hat{P}) \in \Gamma(\flat^*O)$, we note that, for $V' \subseteq_{\mathbf{V}} V$,

$$\begin{aligned}
(\delta_j(\hat{P}))_{V'} &= \delta(\hat{P})_{\flat(V')} \\
&= \delta(\delta(\hat{P})_{\flat(V)})_{\flat(V')} \\
&= O(\flat(V') \hookrightarrow \flat(V))(\delta(\hat{P})_{\flat(V)}) \\
&= (\flat^*O)(V' \hookrightarrow V)(\delta_j(\hat{P})_V).
\end{aligned} \tag{3.15}$$

Because of (3.8) and the fact that $\Gamma(\flat^*O) \subseteq \text{Hyp}_j(\flat^*O)$, $\delta_j(\hat{P})$ can be regarded as a proposition sheaf. That is, it defines elements of $\text{Sub}_{j\text{dB}}(\flat^*O)$ and $\text{Sub}_{j\text{cl}}(\flat^*\Sigma)$ by

$$(\delta_j(\hat{P}))(V) = \{\hat{a} \in (\flat^*O)(V) \mid \hat{a} \preceq (\delta_j(\hat{P}))_V\} \tag{3.16}$$

and

$$(\delta_j(\hat{P}))(V) = \{\sigma \in (\flat^*\Sigma)(V) \mid \sigma((\delta_j(\hat{P}))_V) = 1\}, \tag{3.17}$$

respectively.

As previously noted, every proposition is represented by a clopen subsheaf of $\flat^*\Sigma$. We can assign to it a truth-value, a global element of Ω_j , if we are given a collection of truth propositions. It is internalized as a truth sheaf \mathbb{T}_j , which is a subsheaf of $\mathbb{P}_{j\text{cl}}(\flat^*\Sigma)$ that satisfies appropriate properties. We regard a subsheaf \mathbb{T}_j of $\mathbb{P}_{j\text{cl}}(\flat^*\Sigma)$ as a truth sheaf if and only if $\mathbb{T}_j(V)$ is a filter for every $V \in \mathbf{V}$. That is, if $\mathbb{T}_j(V)$ contains $A \in \text{Sub}_{j\text{cl}}(\flat^*(\Sigma_{\downarrow V}))$ as an element, then it does also any $B \in \text{Sub}_{j\text{cl}}(\flat^*(\Sigma_{\downarrow V}))$ such that $A \subseteq B$. Also, if $A, B \in \mathbb{T}_j(V)$, then $A \cap B \in \mathbb{T}_j(V)$.

Let τ_j be the characteristic morphism of \mathbb{T}_j as a subsheaf of $\mathbb{P}_{j\text{cl}}(\flat^*\Sigma)$. That is, the morphism $\mathbb{P}_{j\text{cl}}(\flat^*\Sigma) \xrightarrow{\tau_j} \Omega_j$ makes the diagram

$$\begin{array}{ccc}
\mathbb{T}_j & \xrightarrow{!} & 1 \\
\downarrow & & \downarrow \text{true}_j \\
\mathbb{P}_{j\text{cl}}(\flat^*\Sigma) & \xrightarrow{\tau_j} & \Omega_j
\end{array} \tag{3.18}$$

a pullback. The morphism τ_j is given by, for each $S \in \text{Sub}_{j\text{cl}}(\flat^*(\Sigma_{\downarrow V}))$,

$$(\tau_j)_V(S) = \{V' \subseteq_{\mathbf{V}} V \mid \flat^*(S_{\downarrow V'}) \in \mathbb{T}_j(V')\}. \tag{3.19}$$

Given a truth sheaf \mathbb{T}_j , we can assign to each proposition P a truth value $\nu(P; \mathbb{T}_j) \in \Gamma\Omega_j$ as

$$\nu(P; \mathbb{T}_j) = \tau_j \circ [P]_j, \quad (3.20)$$

each V -element of which is given by

$$\nu(P; \mathbb{T}_j)_V = \{V' \subseteq_{\mathbf{V}} V \mid \flat^*(P_{\downarrow V'}) \in \mathbb{T}_j(V')\}. \quad (3.21)$$

Let ρ be a density matrix and $r \in [0, 1]$. Döring and Isham [15] defined generalized truth objects $\mathbb{T}_j^{\rho, r}$, the definition of which is given by (A.21). Their global elements represent propositions that are only true with probability at least r in the state ρ . Following Döring and Isham, we define $\mathbb{T}_j^{\rho, r} \in \text{Sub}_{j\text{dB}}(\flat^*O)$ by, for each $V \in \mathbf{V}$,

$$\mathbb{T}_j^{\rho, r}(V) := \{A \in \text{Sub}_{j\text{dB}}(\flat^*(O_{\downarrow V})) \mid \forall V' \subseteq_{\mathbf{V}} V \ (\text{tr}(\rho(\vee A(V')))) \geq r\}. \quad (3.22)$$

It is easy to see that every $\mathbb{T}_j^{\rho, r}(V)$ is a filter, and as we will see in proposition 3.2, it is really a j -sheaf.

When $r = 1$, $\mathbb{T}_j^{\rho, 1}$ gives propositions that are true in the state ρ . Further, when $\rho = |\varphi\rangle\langle\varphi|$, $\mathbb{T}_j^{|\varphi\rangle\langle\varphi|, 1} \equiv \mathbb{T}_j^{|\varphi\rangle\langle\varphi|, 1}$, the counterpart of (A.23), is given by

$$\begin{aligned} \mathbb{T}_j^{|\varphi\rangle\langle\varphi|}(V) &:= \{A \in \text{Sub}_{j\text{dB}}(\flat^*(O_{\downarrow V})) \mid \forall V' \subseteq_{\mathbf{V}} V \ (\delta_j(|\varphi\rangle\langle\varphi|)_{V'} \in A(V'))\} \\ &\simeq \{h \in \text{Hyp}_j(\flat^*(O_{\downarrow V})) \mid \forall V' \subseteq_{\mathbf{V}} V \ (\delta_j(|\varphi\rangle\langle\varphi|)_{V'} \preceq \hat{h}_{V'})\} \\ &= \{h \in \text{Hyp}_j(\flat^*(O_{\downarrow V})) \mid \forall V' \subseteq_{\mathbf{V}} V \ (|\varphi\rangle\langle\varphi| \preceq \hat{h}_{V'})\} \\ &\simeq \{S \in \text{Sub}_{j\text{cl}}(\flat^*(\Sigma_{\downarrow V})) \mid \forall V' \subseteq_{\mathbf{V}} V \ (|\varphi\rangle\langle\varphi| \preceq (\flat^*\alpha)_{V'}^{-1}(S(V')))\}. \end{aligned} \quad (3.23)$$

Proposition 3.2 *For every state ρ and every coefficient $r \in [0, 1]$, $\mathbb{T}_j^{\rho, r}$ is a j -sheaf.*

Proof. First, we show that $\mathbb{T}_j^{\rho, r}$ is a presheaf. To do so, for each $V \in \mathbf{V}$, let $V' \subseteq_{\mathbf{V}} V$ and $V'' \subseteq_{\mathbf{V}} V'$. Then, since we have $\flat(V'') \subseteq_{\mathbf{V}} V'$, it follows that for any $A \in \text{Sub}_{j\text{dB}}(\flat^*(O_{\downarrow V}))$,

$$\flat^*(A_{\downarrow V'})(V'') = A_{\downarrow V'}(\flat(V'')) = A(\flat(V'')), \quad (3.24)$$

hence,

$$\text{tr}(\rho(\vee \flat^*(A_{\downarrow V'})(V''))) = \text{tr}(\rho(\vee A(\flat(V'')))). \quad (3.25)$$

This means that $A \in \mathbb{T}_j^{\rho, r}(V)$ implies $\flat^*(A_{\downarrow V'}) \in \mathbb{T}_j^{\rho, r}(V')$, that is, $\mathbb{T}_j^{\rho, r}$ is a presheaf.

Next, let $A \in (\flat^*\mathbb{T}_j^{\rho, r})(V) = \mathbb{T}_j^{\rho, r}(\flat(V))$; that is, suppose that for every $V' \subseteq \flat(V)$, $\text{tr}(\rho(\vee A(V'))) \geq r$. Then, for every $V' \subseteq_{\mathbf{V}} V$, since $\flat(V') \subseteq_{\mathbf{V}} \flat(V)$, we have

$$\text{tr}(\rho(\vee A(V'))) = \text{tr}(\rho(\vee A(\flat(V')))) \geq r, \quad (3.26)$$

which means $A \in \mathbb{T}_j^{\rho, r}(V)$. Thus, $(\flat^*\mathbb{T}_j^{\rho, r})(V) \subseteq \mathbb{T}_j^{\rho, r}(V)$ results.

Finally, $\mathbb{T}_j^{\rho, r} \xrightarrow{\zeta_{\mathbb{T}_j^{\rho, r}}} \flat^*\mathbb{T}_j^{\rho, r}$ turns out to be a natural isomorphism, because for every $A \in \mathbb{T}_j^{\rho, r}(V)$, $(\zeta_{\mathbb{T}_j^{\rho, r}})_V(A) = \flat^*(A_{\downarrow \flat(V)}) = \flat^*(A_{\downarrow V}) = A$. \square

Let $\mathbb{P}_{j \text{ dB}}(\flat^*O) \xrightarrow{\tau_j^{\rho, r}} \Omega_j$ be the characteristic morphism of $\mathbb{T}_j^{\rho, r}$. From (3.19), we have, for each $A \in \text{Sub}_{j \text{ dB}}(\flat^*(O_{\downarrow V})) = (\mathbb{P}_{j \text{ dB}}(\flat^*O))(V)$,

$$\begin{aligned} (\tau_j^{\rho, r})_V(A) &= \{V' \subseteq_{\mathbf{V}} V \mid \forall V'' \subseteq_{\mathbf{V}} V' \ (\text{tr}(\rho(\vee \flat^*(A_{\downarrow V'})(V''))) \geq r)\} \\ &= \{V' \subseteq_{\mathbf{V}} V \mid \forall V'' \subseteq_{\mathbf{V}} V' \ (\text{tr}(\rho(\vee A(V''))) \geq r)\}. \end{aligned} \quad (3.27)$$

Therefore, the truth-value of a physical proposition $\delta_j(\hat{P})$ corresponding to a projection operator \hat{P} under the truth sheaf $\mathbb{T}_j^{\rho, r}$ is given by, for each $V \in \mathbf{V}$,

$$\begin{aligned} \nu_j(\delta_j(\hat{P}); \mathbb{T}_j^{\rho, r})_V &= \{V' \subseteq_{\mathbf{V}} V \mid \forall V'' \subseteq_{\mathbf{V}} V' \ (\text{tr}(\rho(\delta_j(\hat{P})_{V''})) \geq r)\} \\ &= \{V' \subseteq_{\mathbf{V}} V \mid \text{tr}(\rho(\delta_j(\hat{P})_{V'})) \geq r\}. \end{aligned} \quad (3.28)$$

In particular, for $\mathbb{T}_j^{\rho, r} = \mathbb{T}_j^{|\varphi\rangle}$, we have

$$\begin{aligned} \nu_j(\delta_j(\hat{P}); \mathbb{T}_j^{|\varphi\rangle})_V &= \{V' \subseteq_{\mathbf{V}} V \mid \langle \varphi | (\delta_j(\hat{P})_{V'}) | \varphi \rangle = 1\} \\ &= \{V' \subseteq_{\mathbf{V}} V \mid |\varphi\rangle\langle \varphi | \preceq \delta_j(\hat{P})_{V'}\} \\ &= \{V' \subseteq_{\mathbf{V}} V \mid \delta_j(|\varphi\rangle\langle \varphi |) \in \delta_j(\hat{P})(V')\}. \end{aligned} \quad (3.29)$$

4 Translation Rules of Propositions, Truth Objects, and Truth-Values

In Section 3, we gave the truth-value function ν_j that assigns a truth-value to each proposition sheaf P_j under a given truth sheaf \mathbb{T}_j . In this and the next sections, we clarify the structural relationship between the present sheaf-based theory and the presheaf-based one. What we show in this section

is that, for each P_j and \mathbb{T}_j , there are corresponding proposition presheaves P and truth presheaves \mathbb{T} that can be regarded as ‘translations’, and that there exists a specific relation between global elements of Ω_j and Ω , which is satisfied by $\nu_j(P_j; \mathbb{T}_j)$ and $\nu(P; \mathbb{T})$ for all such propositions P_j and P and truth objects \mathbb{T}_j and \mathbb{T} . Precisely, we show that they satisfy the following relation:

$$\nu_j(P_j; \mathbb{T}_j) = r \circ \nu(P; \mathbb{T}), \quad (4.1)$$

where the morphism r is defined by the epi-mono factorization of j ,

$$\begin{array}{ccc} \Omega & \xrightarrow{j} & \Omega, \\ & \searrow r & \nearrow \\ & \Omega_j & \end{array} \quad (4.2)$$

that is, r is defined by $r_V(\omega) \equiv j_V(\omega) \in \Omega_j(V)$. In the following, we give concrete translation relationships for proposition objects P and P_j and for truth objects \mathbb{T} and \mathbb{T}_j .

First, we give a definition of translation of propositions. Note that each proposition presheaf $P \in \text{Sub}_{\text{dB}}(O)$ is sheafified to a proposition sheaf $\flat^*P \in \text{Sub}_{j\text{dB}}(\flat^*O)$. Therefore, it is quite natural to regard P and P_j as each other’s translation if they satisfy

$$\flat^*P = P_j. \quad (4.3)$$

The following proposition, which is clear from the definition of $\delta_j(\hat{P})$, would suggest (4.3) as a sound definition of translation.

Proposition 4.1 *For every projection operator \hat{P} , $\delta(\hat{P})$ and $\delta_j(\hat{P})$ are each other’s translations.*

Next, we define translation of truth objects. First, we note that, for each truth sheaf $\mathbb{T}_j \in \text{Sub}_j(\mathbb{P}_{j\text{dB}}(\flat^*O))$, the morphism $\flat^*(\mathbb{P}_{\text{dB}}O) \xrightarrow{\varrho_O} \mathbb{P}_{j\text{dB}}(\flat^*O)$ induces a subsheaf of $\flat^*(\mathbb{P}_{\text{dB}}O)$, which we denote by $\varrho_O^{-1}(\mathbb{T}_j)$, as the pullback of \mathbb{T}_j along the morphism ϱ_O ; that is,

$$\begin{aligned} \varrho_O^{-1}(\mathbb{T}_j)(V) &:= (\varrho_O)_V^{-1}(\mathbb{T}_j(V)) \\ &= \{A \in (\flat^*(\mathbb{P}_{\text{dB}}O))(V) \mid \flat^*A \in \mathbb{T}_j(V)\} \\ &= \{A \in \text{Sub}_{\text{dB}}(O_{\downarrow \flat(V)}) \mid \flat^*A \in \mathbb{T}_j(V)\}. \end{aligned} \quad (4.4)$$

On the other hand, each truth presheaf $\mathbb{T} \in \text{Sub}(\mathbb{P}_{\text{dB}}O)$, for which we propose that $\mathbb{T}(V)$ is a filter for every $V \in \mathbf{V}$, has its sheafification $\flat^*\mathbb{T} \in \text{Sub}_j(\flat^*(\mathbb{P}_{\text{dB}}O))$. We say that \mathbb{T} and \mathbb{T}_j are each other's translation, if they satisfy

$$\flat^*\mathbb{T} = \varrho_O^{-1}(\mathbb{T}_j). \quad (4.5)$$

To show soundness of the definition (4.5) of translation, we give the following proposition.

Proposition 4.2 *For every density matrix ρ and $r \in [0, 1]$, the corresponding truth presheaf $\mathbb{T}^{\rho, r}$ and the truth sheaf $\mathbb{T}_j^{\rho, r}$ are each other's translation.*

Proof. Let $A \in \text{Sub}_{\text{dB}}(O_{\downarrow \flat(V)})$ and $h \in \text{Hyp}(O_{\downarrow \flat(V)})$ be its corresponding hyper-element. Then $A \in (\flat^*\mathbb{T}^{\rho, r})(V)$ if and only if

$$\forall V' \subseteq_{\mathbf{V}} \flat(V) \quad \text{tr}(\rho \hat{h}_{V'}) \geq r, \quad (4.6)$$

whereas $A \in (\varrho_O^{-1}(\mathbb{T}_j^{\rho, r}))(V)$ if and only if

$$\forall V' \subseteq_{\mathbf{V}} V \quad \text{tr}(\rho \hat{h}_{\flat(V')}) \geq r. \quad (4.7)$$

What we have to prove is that (4.6) and (4.7) are equivalent.

Suppose that (4.6) holds. Then, since for $V' \subseteq_{\mathbf{V}} V$, we have $\flat(V') \subseteq_{\mathbf{V}} \flat(V)$, (4.7) follows.

Conversely, suppose that (4.7). Then, in particular,

$$\text{tr}(\rho \hat{h}_{\flat(V)}) \geq r. \quad (4.8)$$

On the other hand, for every $V' \subseteq_{\mathbf{V}} \flat(V)$,

$$\hat{h}_{\flat(V)} \preceq \delta(\hat{h}_{\flat(V)})_{V'} \preceq \hat{h}_{V'}. \quad (4.9)$$

Thus, we have

$$r \leq \text{tr}(\rho \hat{h}_{\flat(V)}) \leq \text{tr}(\rho \hat{h}_{V'}), \quad (4.10)$$

which implies (4.6). \square

Now, let P and \mathbb{T} be arbitrary translations of P_j and \mathbb{T}_j , respectively. In the following, we prove that they really satisfy (4.1).

First, note that the names $\lceil P \rceil$ and $\lceil P_j \rceil_j$ make the diagram

$$\begin{array}{ccc}
 & \mathbb{P}_{\text{dB}} O & \\
 \nearrow \lceil P \rceil & \downarrow \zeta_{\mathbb{P}_{\text{dB}} O} & \\
 1 & \mathfrak{b}^*(\mathbb{P}_{\text{dB}} O) & \\
 \searrow \lceil P_j \rceil_j & \downarrow \varrho_O & \\
 & \mathbb{P}_{j \text{ dB}}(\mathfrak{b}^* O) &
 \end{array} \tag{4.11}$$

commute. Here, the definition of $\mathfrak{b}^*(\mathbb{P}_{\text{dB}} O) \xrightarrow{\varrho_O} \mathbb{P}_{j \text{ dB}}(\mathfrak{b}^* O)$, the restriction of $\mathfrak{b}^*(\mathbb{P} O) \xrightarrow{\varrho_O} \mathbb{P}_j(\mathfrak{b}^* O)$, is given in appendix B. The commutativity of (4.11) is easily shown as

$$\begin{aligned}
 (\lceil P_j \rceil_j)_V &= \mathfrak{b}^*((\mathfrak{b}^* P)_{\downarrow V}) \\
 &= \mathfrak{b}^*(P_{\downarrow V}) \\
 &= \mathfrak{b}^*((P_{\downarrow V})_{\downarrow V}) \\
 &= \mathfrak{b}^*((P_{\downarrow V})_{\downarrow \mathfrak{b}(V)}) \\
 &= (\varrho_O)_V((\zeta_{\mathbb{P}_{\text{dB}} O})_V(\lceil P \rceil_V)),
 \end{aligned} \tag{4.12}$$

where we used (B.3) and (B.4).

Proposition 4.3 *Let $\mathbb{P}_{\text{dB}} O \xrightarrow{\tau} \Omega$ and $\mathbb{P}_{j \text{ dB}}(\mathfrak{b}^* O) \xrightarrow{\tau_j} \Omega_j$ be the characteristic morphisms of \mathbb{T} and \mathbb{T}_j , respectively. Then, the diagram*

$$\begin{array}{ccc}
 \mathfrak{b}^*(\mathbb{P}_{\text{dB}} O) & \xrightarrow{\mathfrak{b}^* \tau} & \mathfrak{b}^* \Omega \\
 \downarrow \varrho_O & & \downarrow \mathfrak{b}^* r \\
 & & \mathfrak{b}^* \Omega_j \\
 & & \uparrow \zeta_{\Omega_j} \\
 \mathbb{P}_{j \text{ dB}}(\mathfrak{b}^* O) & \xrightarrow{\tau_j} & \Omega_j
 \end{array} \tag{4.13}$$

commutes if and only if equation (4.5) is satisfied.

Proof. First, we note that, for each $A \in \mathfrak{b}^*(\mathbb{P}_{\text{dB}} O)(V) = \text{Sub}_{\text{dB}}(O_{\downarrow \mathfrak{b}(V)})$,

$$\begin{aligned}
 (\mathfrak{b}^* r \circ \mathfrak{b}^* \tau)_V(A) &= \{V' \subseteq_{\mathbf{V}} \mathfrak{b}(V) \mid \mathfrak{b}(V') \in \tau_{\mathfrak{b}(V)}(A)\} \\
 &= \{V' \subseteq_{\mathbf{V}} \mathfrak{b}(V) \mid A_{\downarrow \mathfrak{b}(V')} \in \mathbb{T}(\mathfrak{b}(V'))\},
 \end{aligned} \tag{4.14}$$

and,

$$(\zeta_{\Omega_j} \circ \tau_j \circ \varrho_O)_V(A) = \{V' \subseteq_{\mathbf{V}} \flat(V) \mid \flat^*(A_{\downarrow V'}) \in \mathbb{T}_j(V')\}. \quad (4.15)$$

Suppose that the diagram (4.13) commutes. Then, for each $V' \subseteq_{\mathbf{V}} \flat(V)$, $A_{\downarrow \flat(V')} \in \mathbb{T}(\flat(V'))$ if and only if $\flat^*(A_{\downarrow V'}) \in \mathbb{T}_j(V')$. In particular, putting $V' = \flat(V)$, we obtain equation (4.5). Conversely, suppose that (4.5) holds. Then, we have, for each $V \in \mathbf{V}$ and $V' \subseteq_{\mathbf{V}} \flat(V)$, $\flat^*\mathbb{T}(V') = \varrho_O^{-1}(\mathbb{T}_j)(V')$; that is, for all $A' \in \text{Sub}(O_{\downarrow \flat(V')})$, $A' \in \mathbb{T}(\flat(V'))$ if and only if $\flat^*A' \in \mathbb{T}_j(V')$. In particular, for any $A \in \text{Sub}_{\text{dB}}(O_{\downarrow \flat(V)})$, we obtain the condition for the diagram (4.13) to commute, by taking $A' = A_{\downarrow \flat(V')}$. \square

To show the relation (4.1), let \mathbb{T} and P be transformations of \mathbb{T}_j and P_j , respectively.

Fitting together (4.11), (4.13), and naturality of ζ , we have a commutative diagram

$$\begin{array}{ccccc}
 & & \mathbb{P}_{\text{dB}}O & \xrightarrow{\tau} & \Omega \\
 & \nearrow [P] & \downarrow \zeta_{\mathbb{P}_{\text{dB}}O} & & \downarrow \zeta_{\Omega} \\
 1 & & \flat^*(\mathbb{P}_{\text{dB}}O) & \xrightarrow{\flat^*\tau} & \flat^*\Omega \\
 & \searrow [P_j]_j & \downarrow \varrho_O & & \downarrow \flat^*r \\
 & & \mathbb{P}_j\text{dB}(\flat^*O) & \xrightarrow{\tau_j} & \Omega_j
 \end{array}
 \quad (4.16)$$

$\nearrow \zeta_{\Omega_j}^{-1}$
 $\searrow \sim$

The outer pentagon of this diagram is just the relation (4.1).

We have proved that for all proposition objects P and P_j satisfying (4.3) and truth objects \mathbb{T} and \mathbb{T}_j satisfying (4.5), the truth-values $\nu(P, \mathbb{T})$ and $\nu_j(P_j, \mathbb{T}_j)$ are related via (4.1). This implies that P and \mathbb{T} represent virtually the same proposition as P_j and the same truth object as \mathbb{T}_j , respectively, from our sheaf-based viewpoint. In this sense, it is reasonable to call them each other's translation. Also, we call the same relation between global elements of Ω_j and Ω as (4.1), that is,

$$\nu_j = r \circ \nu, \quad (4.17)$$

the translation rule of global elements, and say that $\nu_j \in \Gamma\Omega_j$ and $\nu \in \Gamma\Omega$ are each other's translation if they satisfy (4.17).

5 Coarse-Graining Properties of Translation

For a proposition P_j and a truth sheaf and \mathbb{T}_j , their translation presheaves P and \mathbb{T} satisfying (4.3) and (4.5) are not determined uniquely. For such P 's and \mathbb{T} 's, furthermore, the truth-values $\nu(P, \mathbb{T})$ take various values. If we consider their sheaf translations, the various truth-values are transformed to one and the same value $r \circ \nu(P, \mathbb{T})$. In other words, a lot of different propositions, truth objects, and truth-values are not distinguished from the sheaf-based viewpoint. We call this aspect coarse-graining made by translation, the properties of which we observe in the following.

First, let us see coarse-graining of the space $\Gamma\Omega$ of truth-values. The translation rule (4.17) is equivalent to the condition that for all $V \in \mathbf{V}$ and $V' \subseteq_{\mathbf{V}} V$,

$$V' \in (\nu_j)_V \iff \flat(V') \in \nu_V. \quad (5.1)$$

Let $\gamma(\nu_j)$ be the set of all translations $\nu \in \Gamma\Omega$ of ν_j . Note that $\gamma(\nu_j)$ has an order relation \leq inherited from $\Gamma\Omega$. Namely, $\nu_1 \leq \nu_2$ if and only if $(\nu_1)_V \subseteq (\nu_2)_V$ for all $V \in \mathbf{V}$. Furthermore, $\gamma(\nu_j)$ is closed with respect to binary operations on $\Gamma\Omega$, the join \vee and the meet \wedge , each of which is defined by $(\nu_1 \vee \nu_2)_V := (\nu_1)_V \cup (\nu_2)_V$ and $(\nu_1 \wedge \nu_2)_V := (\nu_1)_V \cap (\nu_2)_V$, respectively.

Let us define $\gamma^\vee(\nu_j) \in \Gamma\Omega$ by

$$\gamma^\vee(\nu_j) := 1 \xrightarrow{\nu_j} \Omega_j \twoheadrightarrow \Omega. \quad (5.2)$$

This is the maximum translation of ν_j . In fact, it is clear from the definition (2.10) that (5.1) is satisfied if we put $\nu = \gamma^\vee(\nu_j)$. Furthermore, if $\nu \in \gamma(\nu_j)$, then, since $\flat(V') \in \nu_V$ for every $V' \in \nu_V$, we have $V' \in (\nu_j)_V = \gamma^\vee(\nu_j)_V$ from (5.1). Thus, $\nu \leq \gamma^\vee(\nu_j)$ follows.

Let us define $\gamma^\wedge(\nu_j)$ by

$$\gamma^\wedge(\nu_j)(V) := \{V' \subseteq_{\mathbf{V}} V \mid (\nu_j)_V \cap \mathcal{U}^\flat(V') \neq \emptyset\}, \quad (5.3)$$

where, for each $V \in \mathbf{V}$, $\mathcal{U}^\flat(V)$ is defined by

$$\mathcal{U}^\flat(V) := \{W \in \mathbf{V} \mid V \subseteq_{\mathbf{V}} \flat(W)\}. \quad (5.4)$$

We can straightforwardly verify that $\gamma^\wedge(\nu) \in \gamma(\nu_j)$. Moreover, $\gamma^\wedge(\nu_j)$ is the least translation of ν_j . To see this, let $\nu \in \gamma(\nu_j)$ and $V' \in \gamma^\wedge(\nu_j)_V$. Then, we have $V' \subseteq_{\mathbf{V}} \flat(V'')$ for some $V'' \in (\nu_j)_V$. Furthermore, since for such V'' , $\flat(V'') \in \nu_V$ because of (5.1). Thus, we have $V' \in \nu_V$ since $V' \subseteq_{\mathbf{V}} \flat(V'')$. Conversely, it is easy to show that every $\nu \in \Gamma\Omega$ lying between $\gamma^\wedge(\nu_j)$ and $\gamma^\vee(\nu_j)$ satisfies (5.1). On the other hand, every $\nu \in \Gamma\Omega$ is a translation of $r \circ \nu \in \Gamma\Omega_j$. We thus obtain the following result:

Theorem 5.1 *The truth-value space $\Gamma\Omega$ can be expressed as a disjoint union of the lattices $\gamma(\nu_j)$ ($\nu_j \in \Gamma_j\Omega_j$), each of which is given by*

$$\gamma(\nu_j) = \{\nu \in \Gamma\Omega \mid \gamma^\wedge(\nu_j) \leq \nu \leq \gamma^\vee(\nu_j)\}. \quad (5.5)$$

Next, let us turn to the definition (4.3) of translation of propositions. Let $\mathfrak{z}(P_j)$ be the set of all translation presheaves of P_j . It is clear that $\mathfrak{z}(P_j)$ is an ordered set with respect to the inclusion relation defined on $\text{Sub}_{\text{cl}}\Sigma \simeq \text{Sub}_{\text{dB}}O$. That is, $P_1 \subseteq P_2$ if and only if $P_1(V) \subseteq P_2(V)$ for all $V \in \mathbf{V}$. Also, since (4.3) is equivalent to $P(\flat(V)) = P_j(V)$ for all $V \in \mathbf{V}$, $\mathfrak{z}(P_j)$ is closed for \vee and \wedge defined on $\text{Sub}_{\text{cl}}\Sigma$, where $P_1 \wedge P_2$ and $P_1 \vee P_2$ are defined by $(P_1 \wedge P_2)(V) := P_1(V) \cap P_2(V)$ and $(P_1 \vee P_2)(V) := P_1(V) \cup P_2(V)$, respectively.

Among the translations $P \in \mathfrak{z}(P_j)$, there exists a canonical one $\mathfrak{z}^\vee(P_j)$. To give the definition, we note the following fact.

Proposition 5.2 *If $h \equiv \{\hat{h}_V \in (\flat^*O)(V)\}_{V \in \mathbf{V}}$ is a hyper-element of \flat^*O , it is also a hyper-element of O .*

Proof. Let h be a hyper-element of \flat^*O . Then, since $\flat(V) \subseteq_{\mathbf{V}} V$ for every $V \in \mathbf{V}$, we have $\hat{h}_V \in (\flat^*O)(V) = O(\flat(V)) \subseteq O(V)$, whereas we have $\flat(V') \subseteq_{\mathbf{V}} V'$ for every $V' \subseteq_{\mathbf{V}} V$. Thus, from (3.7) and (3.15), it follows

$$\delta(\hat{h}_V)_{V'} \preceq \delta(\delta(\hat{h}_V)_{V'})_{\flat(V')} = \delta(\hat{h}_V)_{\flat(V')} = \delta_j(\hat{h}_V)_{V'} \preceq \hat{h}_{V'}, \quad (5.6)$$

which means that h is a hyper-element of O . \square

Every proposition sheaf $P_j \in \text{Sub}_{\text{dB}}(\flat^*O)$ has its hyper-element $\{\vee P_j(V)\}_{V \in \mathbf{V}}$ of \flat^*O . We define $\mathfrak{z}^\vee(P_j)$ as the proposition presheaf given by $\{\vee P_j(V)\}_{V \in \mathbf{V}}$ as a hyper-element of O :

$$\begin{aligned} \mathfrak{z}^\vee(P_j)(V) &:= \{\hat{a} \in O(V) \mid \hat{a} \preceq \vee P_j(V)\} \\ &= \{\hat{a} \in O(V) \mid \delta(\hat{a})_{\flat(V)} \in P_j(V)\} \\ &= ((\zeta O)_V)^{-1}(P_j(V)). \end{aligned} \quad (5.7)$$

Clearly, $\mathbf{z}^\vee(P_j)$ satisfies $\mathbf{b}^*(\mathbf{z}^\vee(P_j)) = P$, that is, it is really a translation of P_j .

Proposition 5.3 *For every proposition sheaf $P_j \in \text{Sub}_{j\text{dB}}(\mathbf{b}^*O)$, $\mathbf{z}^\vee(P_j)$ is the largest translation of P_j .*

Proof. The third line of (5.7) means that $\mathbf{z}^\vee(P_j)$ is a pullback of $P_j \rightarrow \mathbf{b}^*O$ along the morphism $O \xrightarrow{\zeta_O} \mathbf{b}^*O$. That is, it makes the tropezoid in the diagram

$$\begin{array}{ccc}
 P & \xrightarrow{\quad} & O \\
 \zeta_P \downarrow & \nearrow \mathbf{z}^\vee(P_j) & \downarrow \zeta_O \\
 \mathbf{b}^*P & \xlongequal{\quad} P_j \rightarrow & \mathbf{b}^*O
 \end{array} \tag{5.8}$$

a pullback. On the other hand, if $\mathbf{b}^*P = P_j$, the outer square commutes because of naturality of ζ . We thus obtain an inclusion $P \rightarrow \mathbf{z}^\vee(P_j)$. \square

For instance, for every projection operator \hat{P} , we have $\delta(\hat{P})_V \preceq \delta(\hat{P})_{\mathbf{b}(V)} = \delta_j(\hat{P})_V$, whereas $\delta_j(\hat{P})$ defines $\mathbf{z}^\vee(\delta_j(\hat{P}))$ as a hyper element of O . Therefore, the proposition presheaf $\delta(\hat{P})$, which is a translation of $\delta_j(\hat{P})$ as previously mentioned, is included by $\mathbf{z}^\vee(\delta_j(\hat{P}))$.

We define $\mathbf{z}^\wedge(P_j)$ by, for each $V \in \mathbf{V}$,

$$\mathbf{z}^\wedge(P_j)(V) := \begin{cases} \{\hat{a} \in O(V) \mid \hat{a} \preceq \bigvee \{\delta(\bigvee P_j(W))_V\}_{W \in \mathcal{U}^b(V)}\} & \text{if } \mathcal{U}^b(V) \neq \emptyset, \\ \{\hat{O}\} & \text{if } \mathcal{U}^b(V) = \emptyset. \end{cases} \tag{5.9}$$

Proposition 5.4 *For every proposition sheaf P_j , $\mathbf{z}^\wedge(P_j)$ is the smallest translation of P_j .*

Proof. Let $k \in \text{Hyp}_j(\mathbf{b}^*O)$ be the hyper-element corresponding to P_j . To show that $\mathbf{z}^\wedge(P_j)$ is a presheaf, we define $h := \{\hat{h}_V \in O(V)\}_{V \in \mathbf{V}}$ by

$$\hat{h}_V := \begin{cases} \bigvee \{\delta(\hat{k}_W)_V\}_{W \in \mathcal{U}^b(V)} & \text{if } \mathcal{U}^b(V) \neq \emptyset, \\ \hat{O} & \text{if } \mathcal{U}^b(V) = \emptyset. \end{cases} \tag{5.10}$$

Since for each V , \hat{h}_V is the top element of $\mathfrak{z}^\wedge(P_j)(V)$, $\mathfrak{z}^\wedge(P_j)$ is a presheaf if and only if h is a hyper-element of O . Let us show this, first.

Suppose that $\mathcal{U}^b(V) \neq \emptyset$. Since $\mathcal{U}^b(V) \subseteq \mathcal{U}^b(V')$ for every $V' \subseteq_{\mathbf{V}} V$, we have

$$\hat{h}_V = \bigvee \{\delta(\hat{k}_W)_V\}_{W \in \mathcal{U}^b(V)} \preceq \bigvee \{\delta(\hat{k}_{W'})_V\}_{W' \in \mathcal{U}^b(V')}. \quad (5.11)$$

On the other hand, since $O(V') \subseteq O(V)$, we have, for every $W' \in \mathcal{U}^b(V')$,

$$\delta(\hat{k}_{W'})_V \preceq \delta(\hat{k}_{W'})_{V'}, \quad (5.12)$$

hence,

$$\bigvee \{\delta(\hat{k}_{W'})_V\}_{W' \in \mathcal{U}^b(V')} \preceq \bigvee \{\delta(\hat{k}_{W'})_{V'}\}_{W' \in \mathcal{U}^b(V')} = \hat{h}_{V'}. \quad (5.13)$$

From (5.11) and (5.13), we have

$$\delta(\hat{h}_V)_{V'} \preceq \delta(\hat{h}_{V'})_{V'} = \hat{h}_{V'}. \quad (5.14)$$

If $\mathcal{U}^b(V) = \emptyset$, then $\delta(\hat{h}_V)_{V'} = \hat{O} \preceq \hat{h}_{V'}$. Thus, h is a hyper-element of O , hence really, $\mathfrak{z}^\wedge(P_j)$ a presheaf.

In order to show that $\mathfrak{z}^\wedge(P_j) \in \mathfrak{z}(P_j)$, it suffices to show that $\hat{h}_{b(V)} = \hat{k}_V$ for every $V \in \mathbf{V}$. Since $V \in \mathcal{U}^b(b(V))$, we have

$$\begin{aligned} \hat{h}_{b(V)} &= \bigvee \{\delta(\hat{k}_W)_{b(V)}\}_{W \in \mathcal{U}^b(b(V))} \\ &= (\delta(\hat{k}_V)_{b(V)}) \vee (\bigvee \{\delta(\hat{k}_W)_{b(V)}\}_{W \in \mathcal{U}^b(b(V)) \setminus \{V\}}). \end{aligned} \quad (5.15)$$

On the other hand, we have $\delta(\hat{k}_V)_{b(V)} = \delta_j(\hat{k}_{b(V)})_{b(V)} = \hat{k}_{b(V)} = \hat{k}_V$, and $\delta(\hat{k}_W)_{b(V)} \preceq \hat{k}_{b(V)} = \hat{k}_V$ for all $W \in \mathcal{U}^b(b(V)) \setminus \{V\}$. Thus, $\hat{h}_{b(V)} = \hat{k}_V$ results.

Finally, to show $\mathfrak{z}^\wedge(P_j)$ to be the smallest translation of P_j , let $l \in \text{Hyp}(O)$ be the hyper-element corresponding to a translation $P \in \mathfrak{z}(P_j)$. What we have to show is $\hat{h}_V \preceq \hat{l}_V$ for all $V \in \mathbf{V}$. It suffices to treat the case where $\mathcal{U}^b(V) \neq \emptyset$. Since we have

$$\delta(\hat{k}_W)_V = \delta(\hat{l}_{b(W)})_V \preceq \hat{l}_V \quad (5.16)$$

for all $W \in \mathcal{U}^b(V)$, it follows that

$$\hat{h}_V = \bigvee \{\delta(\hat{k}_W)_V\}_{W \in \mathcal{U}^b(V)} \preceq \hat{l}_V. \quad (5.17)$$

□

It is obvious that every proposition presheaf $P \in \text{Sub}_{\text{dB}}(O)$ is a translation of P_j if and only if $\mathfrak{z}^\wedge(P_j) \subseteq P \subseteq \mathfrak{z}^\vee(P_j)$. On the other hand, every proposition presheaf P is a translation of the proposition sheaf \flat^*P . Thus, we obtain the following result.

Theorem 5.5 *The proposition space $\text{Sub}_{\text{dB}}(O)$ can be expressed as a disjoint union of the lattices $\mathfrak{z}(P_j)$ ($P_j \in \text{Sub}_{\text{dB}}(\flat^*O)$), each of which is given by*

$$\mathfrak{z}(P_j) = \{P \in \text{Sub}_{\text{cl}}(\Sigma) \mid \mathfrak{z}^\wedge(P_j) \subseteq P \subseteq \mathfrak{z}^\vee(P_j)\}. \quad (5.18)$$

Finally, we observe coarse-graining of truth presheaves. Let $\text{Sub}_{\text{filt}}(\mathbb{P}_{\text{cl}}\Sigma)$ be the set of all truth presheaves; that is, $\mathbb{T} \in \text{Sub}_{\text{filt}}(\mathbb{P}_{\text{cl}}\Sigma)$ means that $\mathbb{T} \in \text{Sub}(\mathbb{P}_{\text{cl}}\Sigma)$ and $\mathbb{T}(V)$ is a filter for every $V \in \mathbf{V}$. We first note that we can define \vee and \wedge on $\text{Sub}_{\text{filt}}(\mathbb{P}_{\text{cl}}\Sigma)$. In fact, we define $\mathbb{T}_1 \vee \mathbb{T}_2 := \mathbb{T}_1 \cap \mathbb{T}_2$, whereas for $\mathbb{T}_1 \wedge \mathbb{T}_2$, we let $(\mathbb{T}_1 \vee \mathbb{T}_2)(V)$ the smallest filter $\mathfrak{F}(\mathbb{T}_1(V) \cup \mathbb{T}_2(V))$ including $\mathbb{T}_1(V) \cup \mathbb{T}_2(V)$, that is,

$$(\mathbb{T}_1 \vee \mathbb{T}_2)(V) := \{P \cap P' \in \text{Sub}_{\text{dB}}(O_{\downarrow V}) \mid P, P' \in \mathbb{T}_1(V) \cup \mathbb{T}_2(V)\}. \quad (5.19)$$

Let $\mathcal{J}(\mathbb{T}_j)$ be the set of all translation presheaves of a truth sheaf \mathbb{T}_j . Since the translation condition (4.5) is equivalent to $\mathbb{T}(\flat(V)) = (\varrho_O)_V^{-1}(\mathbb{T}_j(V))$ for all $V \in \mathbf{V}$, $\mathcal{J}(\mathbb{T}_j)$ is closed for \vee and \wedge defined above.

Also for every truth sheaf \mathbb{T}_j , we can define its canonical translation $\mathcal{J}^\vee(\mathbb{T}_j)$ that is the largest one among the translations satisfying (4.5). It is defined as the pullback of $\varrho_O^{-1}(\mathbb{T}_j) \rightarrow \flat^*(\mathbb{P}_{\text{dB}}O)$ along the morphism $\mathbb{P}_{\text{dB}}O \xrightarrow{\zeta_{\mathbb{P}_{\text{dB}}O}} \flat^*(\mathbb{P}_{\text{dB}}O)$:

$$\begin{aligned} \mathcal{J}^\vee(\mathbb{T}_j)(V) &= \{A \in \text{Sub}_{\text{dB}}(O_{\downarrow V}) \mid (\zeta_{\mathbb{P}O})_V(A) \in \varrho_O^{-1}(\mathbb{T}_j)(V)\} \\ &= \{A \in \text{Sub}_{\text{dB}}(O_{\downarrow V}) \mid A_{\downarrow \flat(V)} \in \varrho_O^{-1}(\mathbb{T}_j)(V)\} \\ &= \{A \in \text{Sub}_{\text{dB}}(O_{\downarrow V}) \mid \flat^*(A_{\downarrow \flat(V)}) \in \mathbb{T}_j(\flat(V))\} \\ &= \{A \in \text{Sub}_{\text{dB}}(O_{\downarrow V}) \mid \flat^*(A_{\downarrow V}) \in \mathbb{T}_j(V)\}. \end{aligned} \quad (5.20)$$

Clearly, if \mathbb{T}_j is a truth sheaf, every $\mathcal{J}^\vee(\mathbb{T}_j)(V)$ is a filter, hence, $\mathcal{J}^\vee(\mathbb{T}_j)$ a truth presheaf.

Next, let us define, for each $V \in \mathbf{V}$ and $W \in \mathcal{U}^b(V)$, $\mathbb{R}_{V;W} \subseteq \text{Sub}_{\text{dB}}(O_{\downarrow V})$ by

$$\mathbb{R}_{V;W} := \{A_{\downarrow V} \in \text{Sub}_{\text{dB}}(O_{\downarrow V}) \mid A \in (\varrho_O^{-1}(\mathbb{T}_j))(W)\}, \quad (5.21)$$

and $\mathbb{R}_V \subseteq \text{Sub}_{\text{dB}}(O_{\downarrow V})$ by

$$\mathbb{R}_V := \bigcup \{\mathbb{R}_V; W\}_{W \in \mathcal{U}^b(V)}. \quad (5.22)$$

We define $\mathbf{j}^\wedge(\mathbb{T}_j)$ by

$$\mathbf{j}^\wedge(\mathbb{T}_j)(V) := \begin{cases} \mathfrak{F}_V(\mathbb{R}_V) & \text{if } \mathcal{U}^b(V) \neq \emptyset \\ \emptyset & \text{if } \mathcal{U}^b(V) = \emptyset, \end{cases} \quad (5.23)$$

where $\mathfrak{F}_V(\mathbb{R}_V)$ is the smallest filter in $\text{Sub}_{\text{dB}}O_{\downarrow V}$ including \mathbb{R}_V .

Proposition 5.6 *For every $\mathbb{T}_j \in \text{Sub}_{\text{filt}}(\mathbb{P}_{\text{cl}}\Sigma)(O)$, $\mathbf{j}^\wedge(\mathbb{T}_j)$ is the smallest translation of \mathbb{T}_j .*

Proof. We prove $\mathbf{j}^\wedge(\mathbb{T}_j)$ to be a presheaf. Suppose that $A \in \mathbf{j}^\wedge(\mathbb{T}_j)(V)$. This is equivalent to that there exists a finite subset \mathbb{S}_V of \mathbb{R}_V such that $\wedge \mathbb{S}_V \subseteq A$, since $\mathbf{j}^\wedge(\mathbb{T}_j)(V)$ is a filter [23]. Therefore, for every $V' \subseteq_{\mathbf{v}} V$, we have $\wedge(\mathbb{S}_V)_{\downarrow V'} \subseteq A_{\downarrow V'}$, where $(\mathbb{S}_V)_{\downarrow V'} \equiv \{B_{\downarrow V'} \mid B \in \mathbb{S}_V\}$ is a finite subset of $\mathbb{R}_{V'}$. Thus, $A_{\downarrow V'} \in \mathbf{j}^\wedge(\mathbb{T}_j)$.

To show that $\mathbf{j}^\wedge(\mathbb{T}_j)$ is a translation of \mathbb{T}_j , note that $V \in \mathcal{U}^b(\mathfrak{b}(V))$. We have, for every $W \in \mathcal{U}^b(\mathfrak{b}(V)) \setminus \{V\}$,

$$\mathbb{R}_{\mathfrak{b}(V); W} = \{A_{\downarrow \mathfrak{b}(V)} \mid A \in (\varrho_O)^{-1}(\mathbb{T}_j)(W)\} \subseteq (\varrho_O)^{-1}(\mathbb{T}_j)(V), \quad (5.24)$$

whereas,

$$\mathbb{R}_{\mathfrak{b}(V); V} = \{A_{\downarrow \mathfrak{b}(V)} \mid A \in (\varrho_O)^{-1}(\mathbb{T}_j)(V)\} = (\varrho_O)^{-1}(\mathbb{T}_j)(V). \quad (5.25)$$

Thus, we obtain

$$\mathbb{R}_{\mathfrak{b}(V)} = \mathbb{R}_{\mathfrak{b}(V); V} \cup \left(\bigcup \{\mathbb{R}_{\mathfrak{b}(V); W}\}_{W \in \mathcal{U}^b(\mathfrak{b}(V)) \setminus \{V\}} \right) = (\varrho_O)^{-1}(\mathbb{T}_j)(V), \quad (5.26)$$

hence,

$$\begin{aligned} \mathbf{j}^\wedge(\mathbb{T}_j)(\mathfrak{b}(V)) &= \mathfrak{F}_{\mathfrak{b}(V)}(\mathbb{R}_{\mathfrak{b}(V)}) \\ &= \mathfrak{F}_{\mathfrak{b}(V)}((\varrho_O)^{-1}(\mathbb{T}_j)(V)) \\ &= (\varrho_O)^{-1}(\mathbb{T}_j)(V), \end{aligned} \quad (5.27)$$

where from the second line to the third, we used the fact that $(\varrho_O)^{-1}(\mathbb{T}_j)(V)$ itself is a filter.

Finally, we show that $\mathcal{J}^\wedge(\mathbb{T}_j)$ is the smallest translation of \mathbb{T}_j . It suffices to show for $V \in \mathbf{V}$ such that $\mathcal{U}^b(V) \neq \emptyset$. Let \mathbb{T} be an arbitrary translation of \mathbb{T}_j . Suppose that $A \in \mathcal{J}^\wedge(\mathbb{T}_j)(V)$. Then, there exists a finite subset \mathbb{S}_V of \mathbb{R}_V such that $\wedge \mathbb{S}_V \subseteq A$. On the other hand, for every $B \in \mathbb{S}_V$, there exists a $W \in \mathcal{U}^b(V)$ such that $B \in \mathbb{R}_{V;W}$; that is, there exists a $C \in (\varrho_O)^{-1}(\mathbb{T}_j)(W) = \mathbb{T}(b(W))$ such that $B = C_{\downarrow V}$. This implies that $B = \mathbb{T}(V \hookrightarrow b(W))(C) \in \mathbb{T}(V)$. Thus, $\mathbb{S}_V \subseteq \mathbb{T}(V)$, hence, $\wedge \mathbb{S}_V \in \mathbb{T}(V)$, which implies $A \in \mathbb{T}(V)$ since $\mathbb{T}(V)$ is a filter. \square

Theorem 5.7 *For every truth sheaf \mathbb{T}_j , $\mathcal{J}(\mathbb{T}_j)$ is a lattice that is given by*

$$\mathcal{J}(\mathbb{T}_j) = \{\mathbb{T} \in \text{Sub}_{\text{filt}}(\mathbb{P}_{\text{cl}}\Sigma) \mid \mathcal{J}^\wedge(\mathbb{T}_j) \subseteq \mathbb{T} \subseteq \mathcal{J}^\vee(\mathbb{T}_j)\}. \quad (5.28)$$

Every truth sheaf \mathbb{T}_j determines a lattice of truth presheaves consisting of translations \mathbb{T}_j . Not all truth presheaves, however, are not translations of truth sheaves. In fact, if \mathbb{T} is a translation of \mathbb{T}_j , $(\rho_O)_V((b^*\mathbb{T})(V)) = \mathbb{T}_j(V)$ needs to be satisfied. However, in general for such \mathbb{T} , it only holds that $(b^*\mathbb{T})(V) \subseteq (\rho_O)_V^{-1}((\rho_O)_V((b^*\mathbb{T})(V)))$. Consequently, the set $\text{Sub}_{\text{filt}}(\mathbb{P}_{\text{cl}}\Sigma)$ of truth presheaves is divided into the pairwise disjoint lattices each of which corresponds to one and the same truth sheaf and the other truth presheaves that fail to be translations.

A Presheaf-Based Truth-Value Assignment

In this appendix, we give a brief explanation of the truth-value assignment method developed by Döring and Isham [15], for the purpose of convenience for comparison with the present truth-value assignment on j -sheaves.

The main ingredient is the spectral presheaf Σ , which is a presheaf such that, for each $V \in \mathbf{V}$, $\Sigma(V)$ is the Gelfand space on V , and for $V' \subseteq_{\mathbf{V}} V$ and $\sigma \in \Sigma(V)$, $\Sigma(V' \hookrightarrow V)(\sigma)$ is a restriction of σ to V' . The spectral presheaf plays a role of state space; every proposition on a given quantum system is assumed to be representable as a clopen subobject S of the spectral presheaf Σ , where S is called a clopen subobject of Σ when $S(V)$ is a closed and open subset of $\Sigma(V)$. Thus, the collection $\text{Sub}_{\text{cl}}(\Sigma)$ of all clopen subobjects of Σ can be regarded as a space of propositions. It is internalized to $\widehat{\mathbf{V}}$ by the clopen power object $\mathbb{P}_{\text{cl}}\Sigma \equiv \Omega^\Sigma$ of Σ , which is expressed as

$$(\mathbb{P}_{\text{cl}}\Sigma)(V) := \text{Sub}_{\text{cl}}(\Sigma_{\downarrow V}), \quad (\text{A.1})$$

and

$$(\mathbb{P}_{\text{cl}}\Sigma)(V' \hookrightarrow V) : \text{Sub}_{\text{cl}}(\Sigma_{\downarrow V}) \rightarrow \text{Sub}_{\text{cl}}(\Sigma_{\downarrow V'}); S \mapsto S_{\downarrow V}. \quad (\text{A.2})$$

There is a bijection from $\text{Sub}_{\text{cl}}(\Sigma)$ to $\Gamma(\mathbb{P}_{\text{cl}}\Sigma) := \text{Hom}(1, \mathbb{P}_{\text{cl}}\Sigma)$ which assigns to each proposition P its name $\llbracket P \rrbracket$ defined by

$$\llbracket P \rrbracket_V := P_{\downarrow V}. \quad (\text{A.3})$$

Here, for each presheaf $Q \in \widehat{\mathbf{V}}$ and $V \in \mathbf{V}$, we define $Q_{\downarrow V} \in \widehat{\mathbf{V}}$ as the downward restriction of Q to $V' \subseteq_{\mathbf{V}} V$:

$$Q_{\downarrow V}(V') := \begin{cases} Q(V') & \text{if } V' \subseteq_{\mathbf{V}} V, \\ \emptyset & \text{otherwise.} \end{cases} \quad (\text{A.4})$$

and for each $V'' \subseteq_{\mathbf{V}} V'$,

$$Q_{\downarrow V}(V'' \hookrightarrow V') := \begin{cases} Q(V'' \hookrightarrow V') & \text{if } V' \subseteq_{\mathbf{V}} V, \\ \emptyset \xrightarrow{!} Q(V'') & \text{otherwise.} \end{cases} \quad (\text{A.5})$$

Döring and Isham gave other ways to express propositions. They are based on the outer presheaf O that is defined by

$$O(V) := \mathcal{P}(V) \quad (\text{A.6})$$

and for $V' \subseteq_{\mathbf{V}} V$,

$$O(V' \hookrightarrow V) : O(V) \rightarrow O(V'); \hat{P} \mapsto \delta(\hat{P})_{V'}. \quad (\text{A.7})$$

Here, $\mathcal{P}(V)$ is the set of all projection operators in V and δ the daseinization operator, which assigns to each projection operator \hat{P} a collection $\delta(\hat{P}) := \{\delta(\hat{P})_V\}_{V \in \widehat{\mathbf{V}}}$, each element $\delta(\hat{P})_V$ of which is defined by

$$\delta(\hat{P})_V := \bigwedge \{\hat{\alpha} \in \mathcal{P}(V) \mid \hat{P} \preceq \alpha\}. \quad (\text{A.8})$$

Obviously, $\delta(\hat{P})$ is a global element of the outer presheaf O . Note that for every $V' \subseteq_{\mathbf{V}} V$, it follows

$$\delta(\delta(\hat{P})_V)_{V'} = \delta(\hat{P})_{V'}. \quad (\text{A.9})$$

This equality is often used in the text.

Döring & Isham proved that

$$\text{Sub}_{\text{dB}}(O) \simeq \text{Hyp}(O) \simeq \text{Sub}_{\text{cl}}(\Sigma), \quad (\text{A.10})$$

and hence for every $V \in \mathbf{V}$,

$$\text{Sub}_{\text{dB}}(O_{\downarrow V}) \simeq \text{Hyp}(O_{\downarrow V}) \simeq \text{Sub}_{\text{cl}}(\Sigma_{\downarrow V}). \quad (\text{A.11})$$

Here, $\text{Sub}_{\text{dB}}(O)$ is the collection of subobjects $B \subseteq O$ such that, for every $V \in \mathbf{V}$, $B(V) \subseteq O(V)$ is a downward closed set of $O(V)$ with a top element. Obviously, it is a complete Boolean lattice. On the other hand, $\text{Hyp}(O)$ is the collection of all hyper-elements of O , where a hyper-element h of O is a collection $\{\hat{h}_V \in O(V)\}_{V \in \mathbf{V}}$ that satisfies, for every $V' \subseteq_{\mathbf{V}} V$,

$$O(V' \hookrightarrow V)(\hat{h}_V) = \delta(\hat{h}_V)_{V'} \preceq \hat{h}_{V'}. \quad (\text{A.12})$$

The bijection relation (A.10) is given by the function $k : \text{Hyp}(O) \xrightarrow{\sim} \text{Sub}_{\text{cl}}(\Sigma)$ defined by

$$(k(h))(V) := \alpha_V(\hat{h}_V), \quad (\text{A.13})$$

and $c : \text{Sub}_{\text{dB}}(O) \xrightarrow{\sim} \text{Hyp}(O)$ defined by

$$c(A)_V := \vee A(V). \quad (\text{A.14})$$

Here, the function $\alpha_V : \mathcal{P}(V) \rightarrow \mathcal{Cl}(\Sigma(V))$, where $\mathcal{Cl}(\Sigma(V))$ is the collection of all clopen subsets of $\Sigma(V)$, is defined as

$$\alpha_V(\hat{P}) := \{\sigma \in \Sigma(V) \mid \sigma(\hat{P}) = 1\}. \quad (\text{A.15})$$

Bijections for (A.11) are given as the restrictions of k and c to subalgebras of V .

In particular, every projection \hat{P} defines a proposition presheaf, the global element $\delta(\hat{P}) \in \Gamma O \subseteq \text{Hyp}(O)$. That is, as an element of $\text{Sub}_{\text{cl}}\Sigma$, $\delta(\hat{P})$ is given by

$$\begin{aligned} (\delta(\hat{P}))(V) &:= \alpha_V(\delta(\hat{P})_V) \\ &= \{\sigma \in \Sigma(V) \mid \sigma(\delta(\hat{P})_V) = 1\}, \end{aligned} \quad (\text{A.16})$$

and as that of $\text{Sub}_{\text{dB}}(O)$,

$$\begin{aligned} (\delta(\hat{P}))(V) &:= c_V^{-1}(\delta(\hat{P})_V) \\ &= \{\hat{a} \in O(V) \mid \hat{a} \preceq \delta(\hat{P})_V\}. \end{aligned} \quad (\text{A.17})$$

Each proposition $P \in \text{Sub}_{\text{cl}}(\Sigma)$ is assigned a truth value relative to a truth object \mathbb{T} , a subobject of $\mathbb{P}_{\text{cl}}\Sigma$ (or, equivalently that of $\mathbb{P}_{\text{dB}}O$) of which global elements give truth propositions. Let τ be the characteristic morphism of \mathbb{T} ; That is, the diagram

$$\begin{array}{ccc} \mathbb{T} & \xrightarrow{!} & 1 \\ \downarrow & & \downarrow \text{true} \\ \mathbb{P}_{\text{cl}}\Sigma & \xrightarrow{\tau} & \Omega \end{array} \quad (\text{A.18})$$

is a pullback. Then, for each proposition P , its truth-value $\nu(P; \mathbb{T}) \in \Gamma\Omega$ is given by

$$\nu(P; \mathbb{T}) = \tau \circ [P], \quad (\text{A.19})$$

or more precisely,

$$\nu(P; \mathbb{T})_V = \{V' \subseteq_{\mathbf{V}} V \mid P_{\downarrow V'} \in \mathbb{T}(V')\}. \quad (\text{A.20})$$

In [15], Döring and Isham defined generalized truth object

$$\mathbb{T}^{\rho, r}(V) := \{A \in \text{Sub}_{\text{dB}}(O_{\downarrow V}) \mid \forall V' \subseteq_{\mathbf{V}} V \ (\text{tr}(\rho(\vee A(V')))) \geq r\}, \quad (\text{A.21})$$

which gives propositions that are true at least a probability $r \in [0, 1]$ in a mixed state expressed by a density matrix ρ . Under the truth presheaf $\mathbb{T}^{\rho, r}$, the truth-value of $\delta(\hat{P})$ is evaluated as

$$\begin{aligned} \nu_j(\delta_j(\hat{P}); \mathbb{T}^{\rho, r})_V &= \{V' \subseteq_{\mathbf{V}} V \mid \forall V'' \subseteq_{\mathbf{V}} V' \ (\text{tr}(\rho(\delta(\hat{P})_{V''})) \geq r)\} \\ &= \{V' \subseteq_{\mathbf{V}} V \mid \text{tr}(\rho(\delta(\hat{P})_{V'})) \geq r\}. \end{aligned} \quad (\text{A.22})$$

If we take $\rho = |\varphi\rangle\langle\varphi|$ and $r = 1$, the truth presheaf $\mathbb{T}^{|\varphi\rangle} := \mathbb{T}^{|\varphi\rangle\langle\varphi|, 1}$ and the truth-value of $\delta(\hat{P})$ are given by

$$\mathbb{T}^{|\varphi\rangle}(V) := \{A \in \mathbb{P}_{\text{dB}}O(V) \mid \forall V' \subseteq_{\mathbf{V}} V \ (\delta(|\varphi\rangle\langle\varphi|)_{V'} \in A(V'))\}, \quad (\text{A.23})$$

and

$$\begin{aligned} \nu(\delta(\hat{P}); \mathbb{T}^{|\varphi\rangle})(V) &= \{V' \subseteq_{\mathbf{V}} V \mid \delta(\hat{P})_{V'} \in \mathbb{T}^{|\varphi\rangle}(V')\} \\ &= \{V' \subseteq_{\mathbf{V}} V \mid \forall V'' \subseteq_{\mathbf{V}} V' \ (\delta(|\varphi\rangle\langle\varphi|)_{V''} \in \delta(\hat{P})(V''))\} \\ &= \{V' \subseteq_{\mathbf{V}} V \mid \forall V'' \subseteq_{\mathbf{V}} V' \ (\delta(|\varphi\rangle\langle\varphi|)_{V''} \preceq \delta(\hat{P})_{V''})\} \\ &= \{V' \subseteq_{\mathbf{V}} V \mid \forall V'' \subseteq_{\mathbf{V}} V' \ (|\varphi\rangle\langle\varphi| \preceq \delta(\hat{P})_{V''})\} \\ &= \{V' \subseteq_{\mathbf{V}} V \mid |\varphi\rangle\langle\varphi| \preceq \delta(\hat{P})_{V'}\}, \end{aligned} \quad (\text{A.24})$$

respectively.

B Mathematical Miscellany

In the text, some propoerties of the functors $\flat : \mathbf{V} \rightarrow \mathbf{V}$ and $\flat^* : \widehat{\mathbf{V}} \rightarrow \widehat{\mathbf{V}}$ are used. In this appendix, we explain them for convenience.

Throughout the text, we use the relation $\flat\flat = \flat$ without notice. Furthermore, the following fact is often used: for any presheaf $Q \in \widehat{\mathbf{V}}$ and any subsheaf A of \flat^*Q ,

$$A = \flat^*A. \quad (\text{B.1})$$

To show this, let $A \in \widehat{\mathbf{V}}$ be a subobject of \flat^*Q . The closure \bar{A} of A in \flat^*Q is given by

$$\begin{aligned} \bar{A}(V) &= \{q \in (\flat^*Q)(V) \mid (\flat^*Q)(\flat(V) \hookrightarrow V)(q) \in A(\flat(V))\} \\ &= \{q \in Q(\flat(V)) \mid q \in A(\flat(V))\} \\ &= A(\flat(V)) \\ &= \flat^*A(V). \end{aligned} \quad (\text{B.2})$$

Thus, A is closed (hence, a sheaf) if and only if $A = \flat^*A$.

When we deal with power objects, the following relations are crucially important: for each $Q \in \widehat{\mathbf{V}}$ and $V \in \mathbf{V}$, we have

$$\flat^*(Q_{\downarrow V}) = \flat^*(Q_{\downarrow \flat(V)}), \quad (\text{B.3})$$

$$\flat^*((\flat^*Q)_{\downarrow V}) = \flat^*(Q_{\downarrow V}), \quad (\text{B.4})$$

and furthermore, for any $V' \subseteq_{\mathbf{V}} V$,

$$\flat^*((\flat^*(Q_{\downarrow V}))_{\downarrow V'}) = \flat^*(Q_{\downarrow V'}). \quad (\text{B.5})$$

They can be proved straightforwardly.

In section 4, we treat a relation between $\flat^*\mathbb{P}$ and $\mathbb{P}_j\flat^*$. They are functors from $\widehat{\mathbf{V}}$ to $\text{Sh}_j\widehat{\mathbf{V}}$, and there exists a canonical natural transformation $\flat^*\mathbb{P} \xrightarrow{\ell} \mathbb{P}_j\flat^*$, which is defined as follows. First, note that, for each presheaf $Q \in \widehat{\mathbf{V}}$ and $V \in \mathbf{V}$,

$$\flat^*(\mathbb{P}Q)(V) = \mathbb{P}Q(\flat(V)) \simeq \text{Hom}(Q_{\downarrow \flat(V)}, \Omega), \quad (\text{B.6})$$

and

$$\mathbb{P}_j(\flat^*Q)(V) \simeq \text{Hom}(\flat^*(Q_{\downarrow V}), \Omega_j) = \text{Hom}(\flat^*(Q_{\downarrow \flat(V)}), \Omega_j). \quad (\text{B.7})$$

Let S be a subobject of $Q_{\downarrow V}$ and $Q_{\downarrow V} \xrightarrow{\chi} \Omega$ be the characteristic morphism of S in $\widehat{\mathbf{V}}$. Then, we have the following commutative diagram:

$$\begin{array}{ccccc}
 & & S & \xrightarrow{!} & 1 \\
 & \swarrow \iota & \downarrow \zeta_S & \searrow \text{true} & \\
 Q_{\downarrow \mathfrak{b}(V)} & \xrightarrow{\chi} & \Omega & & \\
 \downarrow \zeta_{Q_{\downarrow \mathfrak{b}(V)}} & & \downarrow \zeta_\Omega & & \downarrow \\
 \mathfrak{b}^*(Q_{\downarrow \mathfrak{b}(V)}) & \xrightarrow{\mathfrak{b}^*\chi} & \mathfrak{b}^*\Omega & & \mathfrak{b}^*1 \\
 \downarrow \zeta_{\mathfrak{b}^*(Q_{\downarrow \mathfrak{b}(V)})} & & \downarrow \mathfrak{b}^*r & & \downarrow \\
 \mathfrak{b}^*(Q_{\downarrow V}) & \xrightarrow{\mathfrak{b}^*(r \circ \chi)} & \Omega_j & & 1 \\
 & \swarrow \mathfrak{b}^*\iota & \downarrow \zeta_{\Omega_j}^{-1} & \searrow \text{true}_j & \\
 & \mathfrak{b}^*S & \xrightarrow{!} & \mathfrak{b}^*\Omega_j & \xrightarrow{!} 1
 \end{array}
 \tag{B.8}$$

Here, the top square is a pullback, and hence, so is the bottom one, because of the left-exactness of the associated sheaf functor \mathfrak{b}^* . Thus, we define $(\varrho_Q)_V$ as a function that maps the top square to the bottom one; that is, as a function from $\mathfrak{b}^*(\mathbb{P}Q)(V)$ to $(\mathbb{P}_j(\mathfrak{b}^*Q))(V)$, it is defined by

$$(\varrho_Q)_V(S) := \mathfrak{b}^*S, \tag{B.9}$$

and hence, as a function from $\text{Hom}(Q_{\downarrow \mathfrak{b}(V)}, \Omega)$ to $\text{Hom}(\mathfrak{b}^*(Q_{\downarrow \mathfrak{b}(V)}), \Omega_j)$,

$$(\varrho_Q)_V(\chi) := \mathfrak{b}^*(r \circ \chi). \tag{B.10}$$

We can straightforwardly show that $(\varrho_Q)(V)$ is natural with respect to $Q \in \widehat{\mathbf{V}}$ and $V \in \mathbf{V}$.

In the text, the case where Q is the outer presheaf O is treated. As easily shown, the restriction of ϱ_O to $\mathfrak{b}^*(\mathbb{P}_{\text{dB}}O)$ takes values on $\mathbb{P}_{j \text{ dB}}(\mathfrak{b}^*O)$ and the

diagram

$$\begin{array}{ccc}
 \mathfrak{b}^*(\mathbb{P}_{\text{dB}}O) & \xrightarrow{\varrho_O|_{\mathfrak{b}^*(\mathbb{P}_{\text{dB}}O)}} & \mathbb{P}_{j\text{ dB}}(\mathfrak{b}^*O) \\
 \downarrow & & \downarrow \\
 \mathfrak{b}^*(\mathbb{P}Q) & \xrightarrow{\varrho_O} & \mathbb{P}_j(\mathfrak{b}^*Q)
 \end{array} \tag{B.11}$$

commutes. Because of this, in section 4, we write ϱ_O for the restriction $\varrho_O|_{\mathfrak{b}^*(\mathbb{P}_{\text{dB}}O)}$ described above.

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